Ergodic Banach problem, flat polynomials and Mahler's measures with combinatorics

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Abstract. We construct a sequence of polynomials that are flat in the almost everywhere sense. The construction is done by appealing to the nice combinatorial properties of the Singer's sets and Sidon sets. As a consequence, we get a positive answer to Littlewood's flatness problem in the class of the Newman polynomials. We further obtain that there exist a rank one map acting on space of infinite measure with simple Lebesgue spectrum. This answer an old question attributed to Banach. It is turn out that our construction answer also positively the Mahler's problem in the class of Newman polynomials. Moreover, we get an answer to the Bourgain's question on the L^1-norm of L^2 normalized idempotent polynomials.

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1. Introduction

The purpose of this paper is to address the flatness issue in some class of analytic trigonometric polynomials, and to give an alternative proof to the proof given by the author in [**2**]. Therefore, this paper can be seen as a revised version of that article with an alternative proof. But, as in [**2**], the combinatorial Singer's construction and Marcinkiewicz-Zygmund interpolation inequalities [**102**, p.28, chp. X] lies in the heart of the proof. We thus construct, in the same spirit as in [**2**], a class of polynomials that are flat in the almost everywhere sense using the Singer's construction combined with the refinement of the interpolation methods initiated by Marcinkiewicz & Zygmund. It follows that the Hardy spaces and the Carleson interpolation theory play an important role in the proof. For a nice account on the interpolation theory and the H^p theory, we refer the reader to [**40**, p.147, Chap. 9] and [**49**, p.275, Chap. 7], [**57**, p.194], [**91**, p.328].

Our construction further benefited from ideas of Ben Green and Gowers related to the flatness problem in connection with Singer and Sidon sets [**37**], [**52**]. We also take advantage from the recent investigations on the Marcinkiewicz-Zygmund inequalities and its refinements [**33**], [**31**], [**72**], [**83**].

We stress that this paper is deeply indebted to the investigation started in [**3**], [**4**] and [**5**]. So, it is may seen as a companion to those papers.

We further stress that the proof given here is completely different than the proof given in the previous version of this work [**2**]. Although, it was stated in [**2**] that one can give an alternative proof of the main results using a Carlson interpolation theory combined with the methods of disturbed root of unit due to C. Chui and Zhong [**33**]. So, here, our main task is to present this alternative proof .

We remind that the flatness problem was initiated by Littlewood [**70**] and Erdös [**43**], and it has a long history. In the beginning, Littlewood asked on the existence of the sequence of the polynomials on

the circle $P_n(z) = \sum_{j=0}^{n-1} \epsilon_j z^{n_j}$ with $\epsilon_j = \pm 1$ such that

$$A_1\sqrt{n} \leq |P_n(z)| \leq A_2\sqrt{n},$$

where A_1, A_2 are positive absolute constants and uniformly on z of modulus 1. Nowadays the analytic polynomials on the circle with ± 1 coefficients are called Littlewood polynomials.

Erdös and Newman [**42**] considered the problem of the existence of the positive absolute constant A such that

$$\max_{|z|=1}\Big|\sum_{j=0}^{n} a_j z^j\Big| \geq A\sqrt{n},$$

where $|a_j| = 1$. Beller and Newman established that the answer is affirmative if one asked for the polynomials with coefficients bounded by 1 [**17**]. In the opposite, J-P. Kahane disproved the conjecture [**58**].

Besides, using a construction of Byrnes [**25**], T. Körner [**66**] disproved Erdös-Newman's conjecture. But, it is turn out that the main ingredient form [**25**] used by Körner is not valid [**87**]. Nevertheless, Kahane's proof does not used this argument.

Kahane's result has been strengthened by J. Beck who proves that the ultraflat polynomials exist from the class of polynomials of degree n whose coefficients are 400th roots of unity [**15**]. J. Beck's construction is essentially based on the random construction of Kahane.

Since then, it was a long standing problem to obtain effective construction of ultraflat polynomials until solved very recently by Bombieri and Bourgain [**19**]. For a deeper treatment on the Kahane ultraflat polynomials, we refer the reader to [**87**].

The third extremal problem in the class of analytic trigonometric polynomials concern L^1-flatness. This problem seems to be mentioned first in [**79**]. Therein, Newman wrote that it has been conjectured:

Conjecture (Newman [**79**])**.** For any Littlewood polynomial P of degree n, $\big\|P(z)\big\|_1 < c\sqrt{n+1}$, where $c<1$.

In [**80**], Newman solved the problem of L^1-flatness in the class of analytic trigonometric polynomials with coefficients of modulus 1. He proved that the Gauss-Fresnel polynomials are L^1-flat. We refer to [**5**] for a simple proof. This result has been strengthened by Beller [**16**], and Beller & Newman in [**18**] by proving that the sequence of the Mahler measure of the L^2 normalized Gauss-Fresnel polynomials converge to one.

For the polynomials with random coefficients $a_k \in \{+1,-1\}$, Salem and Zygmund [**95**] proved that for all but $o(2^n)$ choices of $a_k=\pm 1$,

$$c_1\sqrt{n\ln(n)} < \Big\|\sum_{k=0}^{n} a_k z^k\Big\|_\infty < c_2\sqrt{n\ln(n)},$$

for some absolute constant $c_1, c_2 > 0$. Halàz [**54**] strengthened this result by proving

$$\Big\|\sum_{k=0}^{n} a_k z^k\Big\|_\infty = \big(1+o(1)\big)C\sqrt{n\ln(n)},$$

For some absolute constant $C>0$. Byrnes and Newman computed L^4-norm of those polynomials [**26**]. Later, Browein & Lokhart [**24**], and Choi & Erdélyi [**35**] used the central limit theorem to compute the limit of the L^p-norm and the Mahler measure of the polynomials with random coefficients ± 1. Their results can be linked to the recent results of Peligrad & Wu [**84**], Barrera & Peligrad, Cohen & Conze [**32**] and Thouvenot & Weiss [**100**]. Therein, the authors investigated a dynamical approach with dynamical coefficients, that is, $a_k = f(T^k x)$, where T is a measure-preserving transformation on some probability space and f is a square-integrable function.

We remind that the polynomials with coefficients $a_k \in \{0,1\}$ and the constant term equal to 1 are nowadays called Newman polynomials. We further notice that the polynomials with coefficients $a_k \in \{0,1\}$ are

known as idempotent polynomials, and since we are concern with L^1-flatness, we may assume that the constant term is 1.

The connection between the Banach problem in ergodic theory and the L^1-flatness problem in the class of Littlewood polynomials or Newman polynomials was established by Bourgain [**20**], Guenais [**50**], and Downarowich & Lacroix [**39**]. M. Guenais proved that the Littlewood problem and the Banach problem are equivalent in some class of dynamical system [**50**]. She further constructed a generalized Fekete polynomials on some torsion groups, and proved that those polynomials are L^1-flat. As a consequence, M. Guenais obtained that there exist a group action with simple Lebesgue component. Subsequently, el Abdalaoui and Lemańczyk proved that the generalized Fekete polynomials constructed by Guenais are ultraflat [**7**]. Very recently, el Abdalaoui and Nadkarni strengthened Guenais's result [**5**] by proving that there exist an ergodic non-singular dynamical system with simple Lebesgue component.

Here, we exhibit a class of L^1-flat polynomials with coefficients 0 and 1. This allow us to produce a dynamical system with simple Lebesgue spectrum. We thus get an affirmative answer to the Banach question.

Furthermore, combining our result with that of [**3**], we provide a positive answer to the Mahler's problem in the class of Newman polynomials [**21**, p.6], [**22**],[**73**].

Our methods breaks down for the polynomials with coefficients ± 1. Thus, we are not able to answer the weaker form of Littlewood question on the existence of L^1-flat polynomials with coefficients ± 1.

We notice that the flatness problem is connected to the number theory and to some practical issues arising in the design of a mobile cellular wireless OFDM system [**89**]. Consequently, it is related to some engineering issues [**39**], [**22**], [**23**].

For the convenience of the reader, we repeat the relevant material from [**3**],[**5**] and [**102**], without proofs, thus making our exposition self-contained.

The paper is organized as follows. In section 2, we state our main results. In section 3, we remind the notion of generalized Riesz products and its connection to ergodic theory. In section 4, we present several definitions of flatness in the class of analytic trigonometric polynomials and the fundamental characterization of L^1-flatness. In section 5, we remind the notion of Singer and Sidon sets in the number theory, and we establish that the L^4-norm strategy can not be apply to the case of the Newman polynomials. Finally, we prove our main results in section 6.

2. Main results

Consider the torus $\mathbb{T} = \Big\{z \in \mathbb{C} \ : \ |z| = 1\Big\}$ equipped with the normalized Lebesgue measure dz. Let $n_0 < n_1 < n_2 < \cdots$ be a positive sequence of integers and put

$$P_n(z) = \sum_{j=0}^{n-1} \epsilon_i \sqrt{p_i} z^{n_i},$$

with $|\epsilon_i| = 1$ and $(p_0, \cdots, p_{n-1})$ is a probability vector. Such polynomials are raised in the study of the spectral type of some class of dynamical systems in ergodic theory. For more details we refer to [**4**] .

Here, we restrict ourself to the case $\epsilon_i = 1$ and $p_i = \frac{1}{n}$. We thus concentred our investigations on the flatness problem in the class of polynomials of the from

$$P_n(z) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} z^{n_i}.$$

Following [**3**], this class is called class B.

We state our main results as follows.

THEOREM 2.1. *There exist a sequence of analytic trigonometric polynomials* $\big(P_n\big)_{n\in\mathbb{N}}$ *with coefficients* 0 *and* 1 *such that the polynomials* $\frac{P_n(z)}{\|P_n\|_2}$ *are flat in almost everywhere sense, that is,*

$$\frac{P_n(z)}{\|P_n\|_2} \xrightarrow[n\to+\infty]{} 1,$$

for almost all z *with respect to the Lebesgue measure* dz*.*

As a consequence, we obtain the following theorem.

THEOREM 2.2. *There exist a dynamical system* $(X, \mathcal{A}, T, \mu)$ *with* $\mu(X) = +\infty$ *and simple Lebesgue spectrum.*

We remind that $(X, \mathcal{A}, T, \mu)$ is a dynamical system if $(X, \mathcal{A}, \mu)$ is a measure space with μ is finite or σ-finite measure, and T is a measure-preserving transformation, that is, for any measurable set A, we have

$$\mu(T^{-1}A) = \mu(A).$$

Theorem 2.2 gives an affirmative answer to the long-standing problem attributed to Banach on the existence of dynamical system which simple Lebesgue spectrum and with no-atomic measure. Let us remind that Ulam in his book [**101**, p.76] stated the Banach problem as follows.

Questions (Banach Problem)**.** Does there exist a square integrable function $f(x)$ and a measure preserving transformation $T(x)$, $-\infty < x < \infty$, such that the sequence of functions $\{f(T^n(x)); n = 1, 2, 3, \cdots\}$ forms a complete orthogonal set in Hilbert space?

The most famous Banach problem in ergodic theory asks if there is a measure preserving transformation on a probability space which has simple Lebesgue spectrum. A similar problem is mentioned by Rokhlin in [**90**]. Precisely, Rokhlin asked on the existence of an ergodic measure preserving transformation on a finite measure space whose spectrum is Lebesgue type with finite multiplicity. Later, Kirillov in his 1966's paper [**61**] wrote "there are grounds for thinking that such examples do not exist". However he has described a measure preserving action (due to M. Novodvorskii) of the group $(\bigoplus_{j=1}^{\infty}\mathbb{Z}) \times \{-1, 1\}$ on the compact dual of discrete rationals whose unitary group has Haar spectrum of

multiplicity 2. Similar group actions with higher finite even multiplicities are also given.

Subsequently, finite measure preserving transformation having Lebesgue component of finite even multiplicity have been constructed by J. Mathew and M. G. Nadkarni [**77**], Kamea [**59**], M. Queffelec [**88**], and
O. Ageev [**10**]. Fifteen years later, M. Guenais produce a torsion group action with Lebesgue component of multiplicity one [**50**].

Our methods is far from making any contribution to this problem. At know, it is seems that this problem is a "dark continent" for the ergodic theory and for the spectral theory of dynamical systems.

Nevertheless, it is turn out that our results allows us to answer a Bourgain's question [**20**] on the supremum of the L^1-norm over all polynomials from class B. Indeed, Theorem 2.1 assert the following.

Corollary 2.3. $\beta = \sup_{n>1} \sup_{k_1<k_2<k_3<\cdots<k_n} \Big\| \frac{1}{\sqrt{n}} \sum_{j=1}^{n} z^{k_j} \Big\|_1 = 1.$

In [**3**], [**11**] and [**34**], the authors established already that $\beta \geq \frac{\sqrt{\pi}}{2}$, and, it easy to see that the simple case $n = 2, k_1 = 0, k_2 = 1$, gives $\beta \geq \frac{2\sqrt{2}}{\pi}$.

We further have.

Corollary 2.4. There exist a sequence of analytic trigonometric polynomials $\big(P_k\big)_{k\in\mathbb{N}}$ with coefficients 0 and 1 such the Mahler measure of the polynomials $\frac{P_k}{\|P_k\|_2}$ converge to 1.

We remind that the Mahler measure of analytic trigonometric polynomials P_k is given by

$$M(P_k) = \exp\Big(\int_{\mathbb{T}} \log\big(\big|P_k(z)\big|\big)dz\Big).$$

Using Jensen's formula [**91**], it can be shown that

$$M(P_k) = \frac{1}{\sqrt{m_k}} \prod_{|\alpha|>1} |\alpha|,$$

where, α denoted the zero of the polynomial $\sqrt{m_k}P_k$. In this definition, an empty product is assumed to be 1 so the Mahler measure of the non-zero constant polynomial $P(x) = a$ is $|a|$. A nice account on the subject may be founded in [**46**, pp.2-11], [**21**].

The next proposition list some elementary properties of the Mahler measure. For the reader's convenience, we provide its proof.

PROPOSITION 2.5. Let $(X, \mathcal{B}, \rho)$ be a probability space. Then, for any two positive functions $f, g \in L^1(X, \rho)$, we have

(i) $M_\rho(f) \stackrel{\text{def}}{=} \exp\Big(\rho(\log(f))\Big)$ is a limit of the norms $||f||_\delta$ as δ goes to 0, that is,

$$||f||_\delta \stackrel{\text{def}}{=} \left(\int f^\delta d\rho\right)^{\frac{1}{\delta}} \xrightarrow[\delta\to 0]{} M_\rho(f),$$

provided that $\log(f)$ is integrable.

(ii) If $\rho\Big\{f > 0\Big\} < 1$ then $M_\rho(f) = 0$.

(iii) If $0 < p < q < 1$, then $\big\|f\big\|_p \leq \big\|f\big\|_q$.

(iv) If $0 < p < 1$, then $M_\rho(f) \leq \big\|f\big\|_p$.

(v) $\displaystyle\lim_{\delta\longrightarrow 0} \int f^\delta d\rho = \rho\Big\{f > 0\Big\}$.

(vi) $M_\rho(f) \leq \big\|f\big\|_1$.

(vii) $M_\rho(fg) = M_\rho(f)M_\rho(g)$.

PROOF. We start by proving (ii). Without loss of generality, assume that $f \geq 0$ and put

$$B = \Big\{f > 0\Big\}.$$

Let $\delta = 1/k$ be in $]0,1[$, $k \in \mathbb{N}^*$. Then $1/(1/\delta) + 1/(1-\delta) = 1/k + (k-1)/k = 1$. Hence, by Hölder inequality, we have

$$\begin{aligned}
\int f^{\delta} d\rho &= \int f^{1/k}.1_B d\rho \\
&\leq \left(\int (f^{1/k})^k dz\right)^{1/k} \left(\int 1_B^{k/k-1} dz\right)^{k-1/k} \\
&\leq \left(\int f d\rho\right)^{1/k} \left(\int 1_B dz\right)^{k-1/k} \\
&\leq \left(\int f d\rho\right)^{1/k} \Big(\rho(B)\Big)^{(k-1)/k}.
\end{aligned}$$

Thus we have proved

$$\begin{aligned}
||f||_{\delta} &\leq \left(\int f d\rho\right) \Big(\rho(B)\Big)^{(1-\delta)/\delta} \\
&\leq \left(\int f d\rho\right) \Big(\rho(B)\Big)^{k-1} \xrightarrow[k \to +\infty]{} 0.
\end{aligned}$$

To prove (i), apply the mean value theorem to the following functions

$$\begin{cases} \delta \longmapsto x^{\delta}, & \text{if } x \in]0,1[; \\ t \longmapsto t^{\delta}, & \text{if } x > 1, \end{cases}$$

Hence, for any $\delta \in]0,1[$ and for any $x > 0$, we have

$$\left|\frac{x^{\delta} - 1}{\delta}\right| \leq x + \Big|\log(x)\Big|.$$

Furthermore, it is easy to see that

$$\frac{f^{\delta} - 1}{\delta} = \frac{e^{\delta \log(f)} - 1}{\delta} \xrightarrow[\delta \to 0]{} \log(f),$$

and, by Lebesgue dominated convergence theorem, we get that

$$\int \frac{f^{\delta} - 1}{\delta} d\rho \xrightarrow[\delta \to 0]{} \int \log(f) d\rho.$$

On the other hand, for any $\delta \in]0,1[$, we have

$$\left|\left|f\right|\right|_\delta = \exp\left(\frac{1}{\delta}\log\left(\int f^\delta d\rho\right)\right),$$

and for a sufficiently small δ, we can write

$$\frac{1}{\delta}\log\left(\int f^\delta d\rho\right) \sim \int \frac{f^\delta - 1}{\delta} d\rho$$

since $\log(x) \sim x-1$ as $x \longrightarrow 1$. Summarizing we have proved

$$\lim_{\delta \longrightarrow 0} ||f||_\delta = exp\left(\int \log(f) d\rho\right) = M_\rho(f).$$

For the proof of (iii) and (iv), notice that the function $x \mapsto x^{\frac{q}{p}}$ is a convex function and $x \mapsto \log(x)$ is a concave function. Applying Jensen's inequality to $\displaystyle\int \left|f\right|^p d\rho$ we get

$$\left\|f\right\|_p \leq \left\|f\right\|_q, \int \log(\left|f\right|) d\rho \leq \log\left(\left\|f\right\|_p\right),$$

and this finishes the proof, the rest of the proof is left to the reader. □

Using the classical Beurling's outer and inner decomposition, el Abdalaoui and Nadkarni in [**3**] computed the Mahler measure of P_k. They further apply the H^p theory to establish a formula for the Mahler measure of the generalized Riesz product. We remind a part of this in the next section.

3. Generalized Riesz products and connection to ergodic theory

The classical notion of Riesz products is based on the notion of dissociation, which can be defined as follows.

Consider the polynomial $P(z) = 1 + z$. Then, we have

$$P(z)^2 = 1 + z + z + z^2.$$

For any integer $N \geq 2$, we can write

$$P(z)P(z^N) = 1 + z + z^N + z^{N+1}.$$

In the first case we group terms with the same power of z, while in the second case all the powers of z in the formal expansion are distinct. In the second case we say that the polynomials $P(z)$ and $P(z^N)$ are dissociated. More generally, we have

LEMMA 3.1 ([**5**]). *If* $P(z) = \displaystyle\sum_{j=-m}^{m} a_j z^j, Q(z) = \sum_{j=-n}^{n} b_j z^j$, $m \leq n$, *are two trigonometric polynomials then for some* N, $P(z)$ *and* $Q(z^N)$ *are dissociated.*

It is well know that if the sequence of polynomials $(|P_j|^2)$ is dissociated (each finite product has dissociation property) with constant term equal to 1. Then, the sequence of probability measures $\Big(\displaystyle\prod_{j=1}^{N} |P_j|^2 dz\Big)$ converge to some probability measure called a Riesz product and denoted by $\displaystyle\prod_{j=1}^{+\infty} |P_j|^2$.

More generally, we have the following definition:

DEFINITION 3.2. Let $P_1, P_2, \cdots$, be a sequence of trigonometric polynomials such that

(i) for any finite sequence $i_1 < i_2 < \cdots < i_k$ of natural numbers

$$\int_{S^1} \Big|(P_{i_1} P_{i_2} \cdots P_{i_k})(z)\Big|^2 dz = 1,$$

where S^1 denotes the circle group and dz the normalized Lebesgue measure on S^1,

(ii) for any infinite sequence $i_1 < i_2 < \cdots$ of natural numbers the weak limit of the measures $\mid (P_{i_1} P_{i_2} \cdots P_{i_k})(z) \mid^2 dz, k = 1, 2, \cdots$ as $k \to \infty$ exists.

Then the measure μ given by the weak limit of $\mid (P_1 P_2 \cdots P_k)(z) \mid^2 dz$ as $k \to \infty$ is called generalized Riesz product of the polynomials $\mid P_1 \mid^2$ $, \mid P_2 \mid^2, \cdots$ and denoted by

$$\mu = \prod_{j=1}^{\infty} \big|P_j\big|^2. \tag{1.1}$$

Connection to ergodic theory and rank one transformations. Using the cut and stack procedure described in [**47**], [**48**], one can construct inductively a family of measure-preserving transformations, called rank one transformations or rank one maps, as follows.

Let B_0 be the unit interval equipped with Lebesgue measure. At stage one we divide B_0 into m_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k^{th} stage we divide the stack obtained at the $(k-1)^{th}$ stage into m_{k-1} equal columns, add spacers and obtain a new stack of height h_k. If during the k^{th} stage of our construction the number of spacers put above the j^{th} column of the $(k-1)^{th}$ stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} < \infty$, $1 \leq j \leq m_k$, then we have

$$h_k = m_{k-1}h_{k-1} + \sum_{j=1}^{m_{k-1}} a_j^{(k-1)}.$$

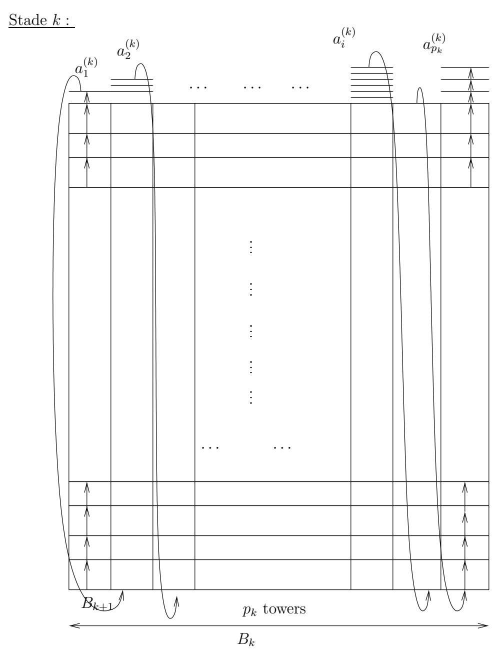

Fig. $(k+1)$th tower.

Proceeding in this way, we get a rank one map T on a certain measure space $(X, \mathcal{B}, |\ .\ |)$ which may be finite or $\sigma-$finite depending on the

number of spacers added.
The construction of a rank one map thus needs two parameters, $(m_k)_{k=0}^{\infty}$ (cutting parameter), and $((a_j^{(k)})_{j=1}^{m_k})_{k=0}^{\infty}$ (spacers parameter). Put

$$T \stackrel{def}{=} T_{(m_k,(a_j^{(k)})_{j=1}^{m_k})_{k=0}^{\infty}}$$

In [**36**] and [**63**] it is proved that the spectral type of this map is given (up to possibly some discrete measure) by

$$d\mu = \mathrm{W}^* \lim \prod_{k=1}^{n} \left|P_k\right|^2 dz, \tag{3.1}$$

where

$$P_k(z) = \frac{1}{\sqrt{m_k}}\left(1 + \sum_{j=1}^{m_k-1} z^{-(jh_k+\sum_{i=1}^{j} a_i^{(k)})}\right),$$

$\mathrm{W}^* \lim$ denotes weak star limit in the space of bounded Borel measures on $\mathbb{T}$.
As mentioned by Nadkarni in [**78**], the infinite product

$$\prod_{l=1}^{+\infty} |P_{j_l}(z)|^2$$

taken over a subsequence $j_1 < j_2 < j_3 < \cdots$, also represents the maximal spectral type (up to discrete measure) of some rank one maps. In case $j_l \neq l$ for infinitely many l, the maps acts on an infinite measure space.

The spectrum of any rank one map is simple and using a random procedure, D. S. Ornstein produced a family of mixing rank one maps [**82**]. It follows that Ornstein's class of maps may possibly contain a candidate for Banach's problem. Unfortunately, in 1993, J. Bourgain proved that almost surely Ornstein's maps have singular spectrum [**20**]. Subsequently, using the same methods, I. Klemes [**62**] showed that the subclass of staircase maps has singular maximal spectral type. In particular, this subclass contains the mixing staircase maps of Adams-Smorodinsky [**8**]. Using a refinement of Peyrière criterium [**86**], I.

Klemes & K. Reinhold proved that the rank one maps have a singular spectrum if the inverse of the cutting parameter is not in ℓ^2 (that is, $\sum_{k=1}^{+\infty} \frac{1}{m_k^2} = +\infty$, where $(m_k) \subset \Big\{2, 3, 4, \cdots\Big\}$ is the cutting parameter) [**63**]. This class contains the mixing staircase maps of Adams & Friedman [**9**]. In 1996, H. Dooley and S. Eigen adapted the Brown-Moran methods [**51**, pp.203-209] and proved that the spectrum of a subclass of Ornstein maps is almost surely singular [**38**].

Later, el Abdalaoui-Parreau and Prikhod'ko extended Bourgain theorem [**20**] by proving that for any family of probability measures in Ornstein type constructions, the corresponding maps have almost surely a singular spectrum [**6**]. They obtained the same result for Rudolph's construction [**92**]. In 2007, el Abdalaoui showed that the spectrum of the rank one map is singular provided that the spacers $(a_j)_{j=1}^{m_k} \subset \mathbb{N}$, are lacunary for all k [**1**]. The author used the Salem-Zygmund central limit theorem methods. As a consequence, the author presented a simple proof of Bourgain's theorem [**20**].

Recently, by appealing to the martingale approximation technique, C. Aistleitner and M. Hofer [**12**] proved a counterpart of the result of [**1**]. Precisely, they proved that the spectrum of the rank one maps is singular provided that the cutting parameter $(m_k) \in \mathbb{N}^*$ and the spacers $(a_j)_{j=1}^{m_k} \subset \mathbb{N}$ satisfy

(i) $\dfrac{\log(m_{k_n})}{h_{k_n}}$ converge to 0;
(ii) the proportion of equal terms in the spacers is at least $c.m_{k_n}$ for some fixed constant c and some subsequence (k_n).

We further recall that I. Klemes & K. Reinhold in [**63**] conjectured that all rank one maps have singular spectrum, and in the same spirit, C. Aistleitner and M. Hofer wrote in the end of their paper "several authors believe that all rank one transformations could have singular maximal spectral type.". It seems that this conjecture was formulated since Baxter result [**14**], [**99**]. We remind that the cutting and stacking rank one construction may goes back to Ornstein's paper in 1960 [**81**]. Indeed, therein, Ornstein constructed a non-singular map for

which there is non σ-finite measure equivalent to Lebesgue measure. Of course, this example is connected to the example of non-singular map with simple Lebesgue component obtain by el Abdalaoui and Nadkarni [**4**]. Notice that in [**82**], the rank one maps are called transformations of class one.

It follows from Bourgain's observation ((eq 2.15)[**20**]) that if the spectral type of any rank one map acting on infinite measure is singular then the spectral type of any rank one is singular. Unfortunately, by our main result, this strategy fails. Therefore, the new approaches are needed to tackle this conjecture.

We remind that in [**3**], it is proved that if $\mu = \displaystyle\prod_{n=1}^{+\infty} |P_n|^2$, then the absolutely continuous part $\frac{d\mu}{dz}$ verify

$$\Big\| \prod_{n=1}^{N} |P_n| - \sqrt{\frac{d\mu}{dz}} \Big\|_1 \xrightarrow[N\to+\infty]{} 0.$$

Furthermore, the Mahler measure of μ [1] satisfy

$$M\Big(\frac{d\mu}{dz}\Big) = \prod_{n=0}^{+\infty} M(P_n^2). \tag{3.2}$$

We further remind from [**3**] the following notion of generalized Riesz products from dynamical origin.

Definition 3.3. A generalized Riesz product $\mu = \displaystyle\prod_{j=1}^{\infty} \big|Q_j(z)\big|^2$, where $Q_j(z) = \displaystyle\sum_{i=0}^{n_j} b_{i,j} z^{r_{i,j}}, b_{i,j} \neq 0, \sum_{i=0}^{n_j} \big|b_{i,j}\big|^2 = 1, \prod_{j=1}^{\infty} \big|b_{n_j,j}\big| = 0$, is said

[1] The Mahler measure of the finite measure μ on the circle is given by

$$M\Big(\frac{d\mu}{dz}\Big) = \inf_P \|P-1\|_{L^2(\mu)},$$

where P ranges over all analytic trigonometric polynomials with zero constant term.

to be of dynamical origin if with

$$h_0 = 1, h_1 = r_{n_1,1} + h_0, \cdots, h_j = r_{n_j,j} + h_{j-1}, j \geq 2$$

it is true that for $j = 1, 2, \cdots$,

$$r_{1,j} \geq h_{j-1},\ r_{i+1,j} - r_{i,j} \geq h_{j-1}.$$

If, in addition, the coefficients $b_{i,j}$ are all positive, then we say that μ is of purely dynamical origin.

The following is proved in [**3**] .

Lemma 3.4. *Given a sequence* $P_n = \displaystyle\sum_{j=0}^{m_n} a_{j,n} z^j, , n = 1, 2, \cdots$ *of analytic trigonometric polynomials in* $L^2(S^1, dz)$ *with non-zero constant terms and* $L^2(S^1, dz)$ *norm 1,* $\displaystyle\prod_{n=1}^{\infty} \left|a_{m_n,n}\right| = 0$. *Then there exist a sequence of positive integers* $N_1, N_2, \cdots$ *such that*

$$\prod_{j=1}^{\infty} \left|P_j(z^{N_j})\right|^2$$

is a generalized Riesz product of dynamical origin.

Applying carefully the previous lemma, the following theorem is proved in [**3**].

Theorem 3.5. *Let* $P_j, j = 1, 2, \cdots$ *be a sequence of non-constant polynomials of* $L^2(S^1, dz)$ *norm 1 such that* $\displaystyle\lim_{j \to \infty} \left|P_j(z)\right| = 1$ *a.e.* (dz) *then there exists a subsequence* $P_{j_k}, k = 1, 2, \cdots$ *and natural numbers* $l_1 < l_2 < \cdots$ *such that the polynomials* $P_{j_k}(z^{l_k}), k = 1, 2, \cdots$ *are dissociated and the infinite product* $\displaystyle\prod_{k=1}^{\infty} \left|P_{j_k}(z^{l_k})\right|^2$ *has finite nonzero value a.e* (dz).

4. flats polynomials

A sequence $P_j, j = 1, 2, \cdots$ of trigonometric polynomials is said to be L^p-flat if the sequence $\frac{|P_j|}{\|P_j\|_2}$, $j = 1, 2, \cdots$ converge to the constant

function 1 in the L^p-norm, $p \in [1, +\infty],\ p \neq 2$. If $p = +\infty$ the sequence P_j is said to be ultraflat.

The flatness issue can be considered for three class of analytic trigonometric polynomials. The polynomials with non-negative coefficients, the Littlewood polynomials which correspond to the polynomials with coefficients ± 1, and the Newman polynomials which correspond to the polynomials with coefficients 0 or 1 and the constant term 1. For all those polynomials, it seems that the existence of L^p-flat polynomials is unknown.

The following notion of almost everywhere flatness is introduced in [**3**].

Definition 4.1. A sequence $P_j, j = 1, 2, \cdots$ of trigonometric polynomials with L^2-norm one is said to be flat almost everywhere, if $P_j(z)$ converge almost everywhere to 1 with respect to dz.

Applying Vitali convergence theorem [**91**] one can see that if P_j is almost everywhere flat then P_j is L^1-flat. In the opposite direction, if P_j is L^1-flat then one can drop a subsequence over which P_j is almost everywhere flat.

We further have the following.

Proposition 4.2. Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of analytic trigonometric polynomials with L^2-norm one. Then, the following are equivalent

(1) $\big(P_n\big)$ is L^1-flat,
(2) $\big(\|P_n\|_1\big)$ converge to 1.

Moreover, if $\big(P_n\big)$ is almost everywhere flat then

$$\||P_n|^2 - 1\|_1 \xrightarrow[n\to+\infty]{} 0.$$

Proof. (1) implies (2) is straightforward. For (2) implies (1), notice that

$$\||P_n| - 1\|_2^2 = 1 = 2\big(1 - \|P_n\|_1\big). \tag{4.1}$$

For the last fact, by Cauchy-Schwarz inequality, we have

$$\||P_n|^2-1\|_1 \leq \||P_n|-1\|_2\||P_n|+1\|_2 \leq 2\||P_n|-1\|_2.$$

The last inequality is due to $\|P_n\|_2 = 1$ combined with the triangle inequality. Thus, it is suffice to see that

$$\||P_n|-1\|_2 \xrightarrow[n\to+\infty]{} 0.$$

But, by (4.1), this equivalent to $(\|P_n\|_1)_{n\in\mathbb{N}}$ converge to 1 which follows from the Vitali convergence theorem. $\square$

We remind that the classical strategy introduced by Newman and Beller to produce the L^1-flat polynomials is based on the reduction of the problem of L^1-flatness to L^4-flatness problem. For the polynomials form class B this strategy fails. This is proved in [**3**]. For sake of completeness, we give the proof in the next section.

5. Sidon sets, Singer sets and flatness

Let R be a positive integer and let $S = \{s_1 < s_2 < s_3 < \cdots < s_R\}$ be a subset of $[0, R)$. Put

$$[S-S]^+ = \Big\{s_j - s_i,\ i<j\Big\} = \Big\{r_1, r_2, \cdots, r_{N(n)}\Big\}.$$

Evidently, $[S-S]^+$ is a subset of $[0, R)$ since

$$0 \leq s_j - s_i \leq s_j < s_R < R.$$

It can be useful to see $[S-S]^+$ as a upper part of the following matrix

$$\mathcal{M}_S = \begin{pmatrix} 0 & s_2-s_1 & s_3-s_1 & \cdots & s_{R-1}-s_1 & s_R-s_1 \\ \cdot & 0 & s_3-s_2 & \cdots & s_{R-1}-s_2 & s_R-s_2 \\ \cdot & \cdot & 0 & s_4-s_3 & \cdots & s_R-s_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & 0 & s_R-s_{R-1} \\ . & . & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

We will denote by $m(l)$ the multiplicity of r_l which correspond to the number of the pair (s_i, s_j) such that $s_j - s_i = r_l$, and we set

$$M(R) = \sup\Big\{m(l), l = 1, \cdots, N(R)\Big\}.$$

Following Chidambaraswamy and Kurtz-Shah [**29**], [**67**], the sequence $\Big\{s_1 < s_2 < s_3 < \cdots < s_R\Big\}$ is δ-admissible if $M(R) = \delta,\ \delta \geq 1$. If $\delta = 1$, then S is called Sidon set. We remind that the classical definition of Sidon sets goes back to Sidon how introduce the notion in 1932 or 1933 according to Erdös [**42**]. This definition can be stated as follows.

Definition 5.1. A set S is called a Sidon set if all the sums $s + t$, $s \leq t \in S$, are distinct.

More generally, one can defined $B_h[g]$ sets, where h and g are a positive integers. A subset A of $[1, N]$ is said to be $B_h[g]$ set if the linear equation $n = \sum_{i=1}^{h} a_i$, has at most g solutions up to permutations. $B_2[1]$ correspond to the Sidon sets. It is easy to see that a subset is Sidon set if and only if all the difference are distinct. This last property is not shared with the $B_h[g]$ sets, $h \geq 3$ [**64**].

Sidon asked on the maximal size of the Sidon set subset of $\{1, \cdots, H\}$. Erdös and Turán [**42**] proved that if $T \subset [0, H]$ is a Sidon set then

$$|T| < \sqrt{H} + 10\sqrt[4]{H} + 1.$$

Lindström strengthened this result and proved [**69**]

$$|T| < \sqrt{H} + \sqrt[4]{H} + 1.$$

In the other direction it has been shown by Chowla [**30**] and Erdös using a theorem of Singer [**97**] that

$$|T| \geq \sqrt{H} - o(\sqrt{H}).$$

Nowadays, it is customary to use algorithmically a Singer's theorem to produce a Sidon subset of the given set $\{1, \cdots, N\}$. Furthermore, the construction can be used to produce a Sidon subset with some desired proprieties of its sumsets [**53**, p.83], [**93**]. We notice that Singer established his theorem in the finite projective geometry setting. In the number theoretic setting, the theorem can be stated as follows.

Theorem 5.2 (Singer [**97**]). *Let p be a prime and let $q = p^2 + p + 1$. Then, there exist $A \subset \mathbb{Z}/q\mathbb{Z}$ with $|A| = p + 1$ such that for all $x \in \mathbb{Z}/q\mathbb{Z} \setminus \{0\}$, there exist a_1, a_2 such that $x = a_1 - a_2$.*

Such set, in which every non-zero difference mod q arises exactly one is called a perfect difference set or Singer set. For the construction of Singer set, we refer the reader to [**97**]. Using the Singer set Erdös-Sárközy-Sós [**44**], [**45**] and Rusza [**93**], [**94**] constructed a Sidon set S subset of $\{1, \cdots, N\}$ such that

$$|S| \geq \sqrt{N} - o(\sqrt{N}) \qquad (ER),$$

With some desired properties.

We notice that Singer's construction is based on the nice properties of finite fields [**97**].

Let us further mention that the lower bound and the upper bound of the quantities $M(R)$ and $N(R)$ can be obtained by considering the following toy examples.

For the first example we take $s_i = i$. This gives

$$P_S(z) = \frac{1}{\sqrt{R}} \sum_{i=1}^{R} z^i,$$

and

$$\Big|P_S(z)\Big|^2 = 1 + \frac{1}{R}\sum_{l=1}^{R}(R-l)z^l + \frac{1}{R}\sum_{l=1}^{R}(R+l)z^{-l}.$$

Therefore $M(R) = R - 1$ and $N(R) = R$. We thus have $M(R)$ is maximal and $N(R)$ is minimal. Indeed, For any $n \in \mathbb{N}^*$, the number of solution of the equation $n = s_j - s_i$ is less than $R - n$ since any solution (s_j, s_i), when it exists, satisfy $n \leq s_j \leq R$.

The second example correspond to the case for which the support of the Fourier transform is a Sidon subset S of $[1, R]$. In this case $M(R) = 1$ and $N(R) = \frac{|S|(|S|-1)}{2}$. Indeed, by definition of the Sidon set all $(s_j - s_i)$ are distinct. Hence, the first row of the matrix $\mathcal{M}_S$ contain $R - 1$ elements, the second row $R - 2$, and the last row one element. By adding, we get

$$(R-1) + (R-2) + \cdots + 1 = \frac{R(R-1)}{2}.$$

Whence $M(R)$ is minimal and $N(R)$ is maximal. It is seems that the quantity $M(R)N(R)$ is balanced.

On L^4-norm strategy and Newman polynomials. It is hidden in the proof given by Chidambaraswamy [**29**] that the L^2-norm of the polynomials $(|P_n(z)|^2 - 1)$ does not converge to 0. Indeed,

$$\Big|P_n(z)\Big|^2 - 1 = \frac{1}{n}\sum_{l=1}^{N(n)} m(l)z^{r_l} + \frac{1}{n}\sum_{l=1}^{N(n)} m(l)z^{-r_l},$$

where $m(l)$ is the multiplicity of r_l given by

$$m(l) = \Big|\Big\{(i,j) \ : s_j - s_i = r_l\Big\}\Big|,$$

and r_l is defined by

$$\Big\{s_j - s_i, \ j < i\Big\} = \Big\{r_1, r_2, \cdots, r_{N(n)}\Big\}.$$

Whence

$$\begin{aligned}\Big\||P_n(z)|^2 - 1\Big\|_2^2 &= \frac{2}{n^2}\sum_{j=1}^{N(n)} m(j)^2 \\ &\geq \frac{2}{n^2}\sum_{j=1}^{N(n)} m(j) \\ &\geq \frac{2}{n^2}\frac{n(n-1)}{2},\end{aligned}$$

since

$$\sum_{j=1}^{N(n)} m(j) = \frac{n(n-1)}{2}.$$

Therefore

$$\Big\||P_n(z)|^2 - 1\Big\|_2^2 \geq 1 + \frac{1}{n},$$

which complete the proof of the claim. From this it easy to see that $\|P_n\|_4 \geq 2$, for any n. Hence, $\|P_n\|_4$ never converge to 1. Thus the L^4-norm strategy of Beller-Newman can not be used.

6. Proofs of the main results

For any finite subset of integer A, we put

$$P_A(z) = \frac{1}{\sqrt{|A|}} \sum_{a \in A} z^a, \;\; z \in \mathbb{T},$$

Where $|A|$ is the number of elements in A. The L^2-norm of P_A is one since

$$\big|P_A(z)\big|^2 = 1 + \frac{1}{|A|} \sum_{\substack{a,b \in A-A \\ a \neq b}} z^{b-a}, \tag{6.1}$$

where $A - A$ is the set of difference of A.
If $|A - A| = |A|^2$ then A is a Sidon set. The nice properties of Singer's set allows us to prove the following.

Lemma 6.1. *Let p be a prime number and S a Singer set of $\mathbb{Z}/q\mathbb{Z}$ with $q = p^2 + p + 1$.. Then for any $r \in \mathbb{Z}/q\mathbb{Z} \setminus \{0\}$, we have*

$$\Big|P_S\big(e^{2\pi i \frac{r}{q}}\big)\Big| = \sqrt{\frac{p}{p+1}}.$$

Proof. Applying (6.1) we get

$$\begin{aligned}
\Big|P_S\big(e^{2\pi i \frac{r}{q}}\big)\Big|^2 &= 1 + \frac{1}{|S|} \sum_{t=1}^{q-1} e^{2\pi i \frac{t.r}{q}} \\
&= 1 - \frac{1}{|S|},
\end{aligned}$$

since

$$\sum_{t=0}^{q-1} e^{2\pi i \frac{t.r}{q}} = 1 + \sum_{t=1}^{q-1} e^{2\pi i \frac{t.r}{q}} = 0.$$

Therefore, we can write

$$\Big|P_S\big(e^{2\pi i \frac{r}{q}}\big)\Big|^2 = \frac{|S| - 1}{|S|} = \frac{p}{p+1},$$

and the proof of the lemma is complete. □

The second main ingredient of our proof is based on the classical Marcinkiewicz-Zygmund inequalities (see [**102**, Theorem 7.5, Chapter X, p.28]) and some ideas linked to its recent refinement obtained by

Chui-Shen-Zhong [**33**] and many others authors. Therefore, by appealing to some classical results form the H^p theory and interpolation theory of Carleson, we will gives an alternative proof to the proof given in [**2**].

As is customary, $[x]$ is the integer part of x, D_n is the Dirichlet kernel, K_n is the Fejér kernel and V_n is the de la Vallé de Poussin kernel. We remind that

$$D_n(x) = \sum_{j=-n}^{n} e^{2\pi i j x} = \frac{\sin\big(\pi(2n+1)x\big)}{\sin(\pi x)},$$

$$K_n(x) = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1} e^{2\pi i j x}\Big) = \frac{1}{n+1}\left\{\frac{\sin\big(\pi(n+1)x\big)}{\sin(\pi x)}\right\}^2,$$

and

$$V_n(x) = 2K_{2n+1}(x) - K_n(x).$$

We remind that the Poisson kernel P_r, $0 < r < 1$, is given by

$$\begin{aligned} P_r(\theta) &= \sum_{-\infty}^{+\infty} r^{|n|} e^{in\theta} \\ &= \frac{1-r^2}{|1-re^{i\theta}|^2} \\ &= \frac{1-r^2}{1-2r\cos(\theta)+r^2}. \end{aligned} \tag{6.2}$$

This kernel is related to the Cauchy kernel $C_r(\theta) \stackrel{\text{def}}{=} \dfrac{1}{1-re^{i\theta}}$ by the following relation

$$P_r = \mathrm{Re}(H_r), \text{ where } H_r = 2C_r - 1.$$

The imaginary part of H_r is called the conjugate Poisson kernel and denoted by

$$Q_r(\theta) = \frac{2r\sin(\theta)}{1-2r\cos(\theta)+r^2}.$$

Let us also remind that if $f = u + i\widetilde{u}$ is analytic in the closed disc with $f(0)$ is real then

$$f(re^{i\theta}) = u * H_r(\theta),$$

and

$$\widetilde{u}(\theta) = u * Q_r(\theta).$$

We notice that $\widetilde{u}$ is the harmonic conjugate to u, which vanishes at the origin, and of course, Q_r is the the harmonic conjugate to P_r. For $f \in L^1(\mathbb{T})$, the harmonic conjugate of f is given by

$$\widetilde{f}(re^{i\theta}) = Q_r * f(\theta) = -i \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{n}{|n|} r^{|n|} \widehat{f}(n) e^{int}.$$

It is well known that the radial limit of $\widetilde{f}(re^{i\theta})$ exist almost everywhere, and this radial limit denoted by $\widetilde{f}$ is the conjugate function of f.

We will use often the following classical property: If $F = \exp(H)$, where H is an analytic function. Then

$$|F| = \exp(\mathrm{Re}(H)).$$

Given a continuous function f on the torus $\mathbb{T}$ and a triangular family of equidistant points $z_{n,j} \in \mathbb{T}, j = 0, \cdots 2n$, $n \in \mathbb{N}^*$, that is,

$$z_{n,j} = z_{n,0} + \frac{j}{2n+1},\ j = 0, \cdots, 2n.$$

We define the Lagrange polynomial interpolation of f at $\{z_n, j\}$ by

$$L_n(f, \{z_n, j\})(e^{2\pi i x}) = \frac{1}{2\pi} \int_0^{2\pi} f(t) D_n(x - z_{n,j}) d\omega_{2n+1}(t),$$

where ω_{2n+1} is a function defined by

$$\omega_{2n+1}(t) = \frac{2\pi j}{2n+1} \ \text{ for } \frac{2\pi j}{2n+1} \leq t < \frac{2\pi(j+1)}{2n+1}, j = 0, \pm 1, \pm 2, \cdots.$$

ω_{2n+1} is a step function with jump $\frac{2\pi}{2n+1}$ at the points $\frac{2\pi j}{2n+1}$ and $d\omega_{2n+1}$ its Riemann-Stieltjes integral. In the same manner, we define the step function ω_m, for any $m \in \mathbb{N}^*$ and we denote its Riemann-Stieltjes integral by $d\omega_m$.

We will need the following classical inequality due to S. Bernstein and A. Zygmund. For its proof, we refer to [**102**, Theorem 3.13, Chapter X, p. 11].

THEOREM 6.2. *[Bernstein-Zygmund inequality]. For any $p \geq 1$, for any polynomial P of degree n, we have*

$$\big\|P'\big\|_p \leq n\big\|P\big\|_p,$$

where P' is the derivative of P. The equality holds if and only if $P(e^{ix}) = M\cos(nx+\xi)$.

Máté, Nevai and Arestov extended Bernstein-Zygmund inequality by proving that it is valid for $p \geq 0$. [**21**, p.142]. For $p = 0$, the result is due to Mahler, a simple proof can be found in [**46**]. Although we will not need this result in such generality.

The Marcinkiewicz-Zygmund interpolation inequalities assert that for $\alpha > 1$, $n \geq 1$, and polynomial P of degree $\leq n-1$,

$$(6.3)\ \frac{A_\alpha}{n}\sum_{j=0}^{n-1}\big|P(e^{2\pi i\frac{j}{q}})\big|^\alpha \leq \int_{\mathbb{T}}\Big|P(z)\Big|^\alpha dz \leq \frac{B_\alpha}{n}\sum_{j=0}^{n-1}\big|P(e^{2\pi i\frac{j}{q}})\big|^\alpha,$$

where A_α and B_α are independent of n and P.

The left hand inequality in (6.3) is valid for any non-negative non-decreasing convex function and in the more general form [**102**, Remark, Chapter X, p. 30]. For sake of completeness, we will state and present a sketch of the proof of it.

THEOREM 6.3. *Let $\kappa > 0$, $m \geq (1+\kappa)2n$. Then, for any non-negative, non-decreasing and convex function ϕ, for any trigonometric polynomial Q of degree n, we have*

$$\int_0^{2\pi}\phi(A_\kappa|Q|)d\omega_m \leq \int_0^{2\pi}\phi(|Q|)dx,$$

where

$$A_\kappa = \frac{1}{1+\kappa^{-1}}.$$

PROOF. The proof is the same as in [**102**, p.29]. One only needs to substitute the de Vallé de Poussin kernel V_{n-1} by

$$V_{n,h} = \Big(1+\frac{n}{h}\Big)K_{n+h-1} - \frac{n}{h}K_{n-1},$$

where $h = [2\kappa n] + 1$ and $\frac{1}{3}$ by $\frac{1}{1+\frac{2n}{h}}$. $V_{n,h}$ is the de Vallé de Poussin kernel of order h. □

Remark. In [**102**, Theorem 7.28, Chapter X, p. 33], one may found the proof of the right hand inequality in the Marcinkiewicz-Zygmund inequalities under the same assumptions as in Theorem 6.3.

The next lemma is crucial for the proof of our main result.

LEMMA 6.4. *Let p be a prime number and S a Singer set of $\mathbb{Z}/q\mathbb{Z}$ with $q = p^2 + p + 1$. Then, for any $\alpha > 1$, we have*

$$\frac{1}{q}\sum_{r=0}^{q-1}\Big|P_S\big(e^{2\pi i\frac{r}{q}}\big)\Big|^{\alpha} = \frac{1}{q}\Big((p+1)^{\frac{\alpha}{2}} + (q-1)\Big(\frac{p}{p+1}\Big)^{\frac{\alpha}{2}}\Big).$$

PROOF. It is straightforward from Lemma 6.1. □

Lemma 6.4 yields for any $0 < \alpha < 4$,

$$\lim_{q\longrightarrow+\infty}\frac{1}{q}\sum_{r=0}^{q-1}\Big|P_S\big(e^{2\pi i\frac{r}{q}}\big)\Big|^{\alpha} = 1. \tag{6.4}$$

Now, following the strategy in [**33**], we perturb the root of unity as follows. Put

$$\begin{aligned} t_{q,r} &= \frac{r}{q}, \text{ and} \\ t^*_{q,r} &= \frac{r}{q} \pm \frac{\delta}{q.p^{1/2+\epsilon}},\ \delta > 0, \epsilon > 0. \end{aligned}$$

We thus have

LEMMA 6.5. *For any $0 < \alpha < 4$,*

$$\lim_{q\longrightarrow+\infty}\frac{1}{q}\sum_{r=0}^{q-1}\Big|P_S\big(e^{2\pi i t^*_{q,r}}\big)\Big|^{\alpha} = 1.$$

PROOF. Applying Bernstein theorem (Theorem 6.2), we get

$$\begin{aligned} \Big|P(e^{2\pi i t_{r,q}}) - P(e^{2\pi i t^*_{q,r,\delta}})\Big| &\leq q\|P_S\|_\infty\Big|e^{2\pi i t_{r,q}} - e^{2\pi i t^*_{q,r,\delta}}\Big| \\ &\leq \frac{\sqrt{p+1}}{2\pi}\frac{\delta}{p^{1/2+\epsilon}} \xrightarrow[q\to\infty]{} 0. \end{aligned}$$

This combined with the standard triangle inequalities gives

$$\begin{aligned}
&\left|\left(\frac{1}{q}\sum_{r=0}^{q-1}\Big|P_S\big(e^{2\pi i t_{q,r}}\big)\Big|^{\alpha}\right)^{\frac{1}{\alpha}}-\left(\frac{1}{q}\sum_{r=0}^{q-1}\Big|P_S\big(e^{2\pi i t^*_{q,r,\delta}}\big)\Big|^{\alpha}\right)^{\frac{1}{\alpha}}\right|\\
\leq\ &\left(\frac{1}{q}\sum_{r=0}^{q-1}\Big|P_S\big(e^{2\pi i t_{q,r}}\big)-P_S\big(e^{2\pi i t^*_{q,r,\delta}}\big)\Big|^{\alpha}\right)^{\frac{1}{\alpha}}\\
\leq\ &\frac{\sqrt{p+1}}{2\pi}\frac{\delta}{p^{1/2+\epsilon}}\xrightarrow[q\to\infty]{}0,
\end{aligned}$$

and the proof of the lemma is complete. □

Lemma 6.5 allow us to construct a new families of nodal points for which (6.4) holds.

Now, we are able to prove our main results.

6.1. Proof of Theorems 2.1, 2.2. In the previous version of this work [**2**], Theorem 2.1 was proved by showing that the L^1-flatness holds by appealing to Lemma 6.5 to construct a disturbed triangle family of nodal points $\{z^*_{q,j}\}_{j=0}^{m-1}$ of $\{z_{q,j}\}$ that satisfy (6.4) and with $m \geq (1+\kappa)2q$, $\kappa \ll q$, where $\kappa > 0$ is given. As a consequence, by Theorem 6.3, we obtain that

$$\lim_{q\longrightarrow+\infty}\big\|P_q\big\|_1 \geq \Big(1+\kappa^{-1}\Big)^{-1}, \tag{6.5}$$

and letting $\kappa \longrightarrow +\infty$, we get

$$\lim_{q\longrightarrow+\infty}\big\|P_q\big\|_1 = 1. \tag{6.6}$$

We notice that there is many way to obtain such sequence of disturbed points. Here, we follows the spirit of the Kadets 1/4 theorem for polynomials due to Marzo-Seip [**76**]. This new strategy is based on the Carleson interpolation theory and the very recent refinement of the Marcinkiewicz-Zygmund inequalities.

We start by proving Theorem 2.1.

Let S be a fixed Singer set in $\mathbb{Z}/q\mathbb{Z}$ with $q = p^2 + p + 1$, p prime number and put

$$P_q(z) = P_S(z).$$

Define

$$z_{j,q} = e^{2\pi i \frac{j}{q}},$$

and for a given $\delta_{q,j} > 0$, $j = 0, \cdots, q-1$, we put

$$z^*_{r,q,(\delta_{q,j})} = e^{2\pi i \left(\frac{j}{q} + \frac{\delta_{q,j}}{q}\right)}.$$

Let $\delta > 0$. We define

$$z_{r,2q,\delta} = \begin{cases} z_{\frac{r}{2},q}, & \text{if } r \text{ is even;} \\ z^*_{\frac{r-1}{2},q,\delta}, & \text{if } r \text{ is odd,} \end{cases}$$

with $\delta_{q,j} = \delta, j = 0, \cdots, q-1$, and $\rho_q = 1 - \frac{1}{2q}$. We thus have

$$\left\{z_{r,2q,\delta}\right\} = \left\{z_{r,q}\right\}_{r=0}^{q-1} \bigcup \left\{z^*_{r,q,\delta}\right\}_{r=0}^{q-1},$$

and we set

$$F_{2q-1}(z) = \prod_{r=0}^{2q-1} \left(1 - \rho_q \overline{z_{r,2q,\delta}} z\right), \text{ where } \rho_q = \frac{2q-1}{2q}.$$

We remind that the family of nodal points $\mathcal{Z} = \left\{\{z_{j,n}\}_{j=0}^n\right\}_{n \geq 0}$ is said to be an L^α Marcinkiewicz-Zygmund family if the L^α Marcinkiewicz-Zygmund inequalities holds for the nodal points $\{z_{j,n}\}_{j=0}^n$, for every $n \geq 0$.

We associate to any family of nodal points $\mathcal{Z} = \left\{\{z_{j,n}\}_{j=0}^n\right\}_{n \geq 0}$ the function F_n defined by

$$F_n(z) = \prod_{r=0}^{n} \left(1 - \rho_n \overline{z_{r,n}} z\right), \text{ where } \rho_n = 1 - \frac{1}{n+1}.$$

The family $\mathcal{Z}$ is said to be uniformly separated if there is a positive number c such that

$$\inf_{j \neq k} \left|z_{n,j} - z_{n,k}\right| \geq \frac{c}{n+1}, \ \forall \ n \geq 0.$$

This notion is related to the notion of Carleson measures and following [**96**] the sequence $\mathcal{X} = \{\xi_n\}$ of points in the open unit disk $\mathbb{D}$ is said to satisfy Carleson's condition if,

$$\gamma = \inf_k \prod_{\substack{j=1\\ j\neq k}}^{\infty} \Big|\frac{\xi_j - \xi_j}{1-\overline{\xi_k}\xi_j}\Big| > 0. \tag{6.7}$$

Of course this condition is connected to the well-known Carleson's interpolation theorem [**28**]. For the proof of Carleson's interpolation theorem, we refer the reader to [**40**, p.157], [**65**, p.1], [**49**, p.274].

We remind that a finite measure μ is a Carleson measure if the injection mapping from H^α, $\alpha > 0$ to the space $L^\alpha(\mathbb{D}, \mu)$ is bounded. These measures were described geometrically in Carlson's theorem [**27**], [**40**, p.156], which assert that the finite measure μ is a Carleson measure if and only if there exist a constant $C_\alpha > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^\alpha d\mu \leq C_\alpha \big\|f\big\|_{H^\alpha}^\alpha \ f \in H^\alpha,$$

for any $\alpha > 0$. Furthermore, by Carleson's interpolation theorem, we have that the discrete measure μ given by

$$\mu = \sum_{n=1}^{+\infty} (1 - |z_k|^2)\delta_{z_k},$$

where δ_w is the Dirac measure on w, is a Carlson measure if the family $\{z_k\}$ is uniformly separated. This result was strengthened in [15] by McDonald and Sundberg [**74**], who proved that the sequence $\{z_k\}$ of points in $\mathbb{D}$ generates a discrete Carleson measure μ if and only if $\{z_k\}$ is a finite union of uniformly separated sequences. For a simple proof, we refer to [**41**]. We notice that if the sequence is uniformly separated then the constant C_α depend uniquely on γ. In this setting, we have the following lemma

Lemma 6.6. *The sequences* $\mathcal{Z} = \Big\{\big\{\rho_q z_{r,q}\big\}\Big\}_{q\geq 0}$ *and* $\mathcal{Z}^* = \Big\{\big\{\rho_q z^*_{r,q,\delta}\big\}\Big\}_{q\geq 0}$ *are uniformly separated sequences.*

Proof. Put

$$\xi_r^* = \rho_q z_{r,q,\delta}^* = \rho_q e^{it_{r,q}},$$

where $t_{r,q} = 2\pi(\frac{r}{q} + \frac{\delta}{q})$, $r = 0, \cdots, q-1$. Then

$$\begin{aligned}
\Big|\frac{\xi_r^* - \xi_s^*}{1 - \overline{\xi_s^*}\xi_r^*}\Big|^2 &= \frac{2\rho_q^2\Big(1 - \cos\big(t_{r,q} - t_{s,q}\big)\Big)}{1 - 2\rho_q^2\cos\big(t_{r,q} - t_{s,q}\big) + \rho_q^4} \\
&= \frac{4\rho_q^2\Big(\sin\big(\frac{t_{r,q}-t_{s,q}}{2}\big)\Big)^2}{\big(1-\rho_q^2\big)^2 + 4\rho_q^2\Big(\sin\big(\frac{t_{r,q}-t_{s,q}}{2}\big)\Big)^2} \\
&= \frac{4\rho_q^2\Big(\sin\big(\pi\frac{r-s}{q}\big)\Big)^2}{\big(1-\rho_q^2\big)^2 + 4\rho_q^2\Big(\sin\big(\pi\frac{r-s}{q}\big)\Big)^2} \qquad (6.8)
\end{aligned}$$

Notice that $\pi.\frac{r-s}{q} \in]-\pi, \pi[$, and the function $x \mapsto \sin^2(x)$ is an even function. We further have, for any $x \in \mathbb{R}$,

$$\sin^2(x - \pi) = \sin^2(x).$$

Therefore, we can reduce our study to the case of $\pi.\frac{r-s}{q} \in]0, \pi/2]$, and if $\pi.\frac{r-s}{q} \in [\pi/2, \pi[$ we substitute $\pi.\frac{r-s}{q}$ by $\pi.\frac{r-s}{q} - \pi \in [-\pi/2, 0[$.
Now, assuming $\pi.\frac{r-s}{q} \in]0, \pi/2]$, it follows that

$$\sin^2\Big(\pi.\frac{r-s}{q}\Big) \geq 4\frac{(r-s)^2}{q^2},$$

since, for any $x \in]0, \pi/2]$, we have $\sin(x) \geq \dfrac{2}{\pi}x$. Whence

$$\frac{4\rho_q^2\Big(\sin\big(\pi\frac{r-s}{q}\big)\Big)^2}{\big(1-\rho_q^2\big)^2 + 4\rho_q^2\Big(\sin\big(\pi\frac{r-s}{q}\big)\Big)^2} \geq \frac{4\rho_q^2\Big(4\frac{(r-s)^2}{q^2}\Big)}{\big(1-\rho_q^2\big)^2 + 4\rho_q^2\Big(4\frac{(r-s)^2}{q^2}\Big)}, \qquad (6.9)$$

by the monotonicity of the function $\phi(x) = \dfrac{4\rho_q^2 x}{\big(1-\rho_q^2\big)^2 + 4\rho_q^2 x}$.
We further have

$$\big(1 - \rho_q^2\big)^2 \leq \frac{8\rho_q^2}{q^2} \leq \frac{16\rho_q^2}{q^2},$$

since, for any $n \geq 1$, $(n-1) \leq 2\sqrt{2}(n-1)$. This combined with (6.8) and (6.9) gives

$$\Big|\frac{\xi_r^* - \xi_s^*}{1 - \overline{\xi_s^*}\xi_r^*}\Big|^2 \geq \frac{(r-s)^2}{1+(r-s)^2}. \tag{6.10}$$

We thus get

$$\inf_s \prod_{\substack{r=0\\ r\neq s}}^{q-1} \Big|\frac{\xi_r^* - \xi_s^*}{1 - \overline{\xi_s^*}\xi_r^*}\Big|^2 \geq \prod_{t=1}^{+\infty}(1 - \frac{1}{1+t^2}) \stackrel{\text{def}}{=} \gamma^2 > 0,$$

by the convergence of $\displaystyle\sum_{t=1}^{+\infty} \frac{1}{1+t^2}$. □

It follows from Lemma 6.6 that the union of the families $\mathcal{Z}$ and $\mathcal{Z}^*$ generates a Carleson measure since the sum of two Carleson measures is a Carlson measure. We further deduce the following

Lemma 6.7. *The sequence* $\mathcal{Z} = \Big\{\big\{\rho_q z_{r,q}\big\}\Big\}_{q\geq 0} \cup \Big\{\big\{\rho_q z^*_{r,q,\delta}\big\}\Big\}_{q\geq 0}$ *is uniformly separated, and we have*

$$\inf_{\xi\in\mathcal{Z}} \prod_{\substack{\chi\in\mathcal{Z}\\ \chi\neq\xi}} \Big|\frac{\chi-\xi}{1-\bar{\xi}\chi}\Big| \geq \gamma^2.\frac{\delta}{\sqrt{1+\delta^2}},$$

where

$$\gamma^2 = \prod_{t=1}^{+\infty}\Big(1 - \frac{1}{1+t^2}\Big).$$

Proof. Put

$$\xi_r = \rho_q z_{r,q}, \text{ and } \xi_r^* = \rho_q z^*_{r,q,\delta},$$

and let $\xi_s \in \mathcal{Z}$, $s = 0, \cdots q-1$. Then, either $\xi_s \in \left\{\rho_q z_{r,q}\right\}$ or $\xi_s \in \left\{\rho_q z^*_{r,q,\delta}\right\}$. Assuming that $\xi_s \in \left\{\rho_q z_{r,q}\right\}$, it follows that

$$\begin{aligned}
\prod_{\substack{\chi \in \mathcal{Z} \\ \chi \neq \xi}} \Big|\frac{\chi - \xi}{1 - \overline{\xi}\chi}\Big| &= \prod_{r \neq s} \Big|\frac{\xi_r - \xi_s}{1 - \overline{\xi_s}\xi_r}\Big| \prod_{r=0}^{q-1} \Big|\frac{\xi^*_r - \xi_s}{1 - \overline{\xi_s}\xi^*_r}\Big| \\
&= \prod_{r \neq s} \Big|\frac{\xi_r - \xi_s}{1 - \overline{\xi_s}\xi_r}\Big| \prod_{r \neq s} \Big|\frac{\xi^*_r - \xi_s}{1 - \overline{\xi_s}\xi^*_r}\Big| \Big|\frac{\xi^*_s - \xi_s}{1 - \overline{\xi_s}\xi^*_s}\Big| \\
&\geq \gamma^2 \frac{\delta}{\sqrt{1+\delta^2}},
\end{aligned}$$

by the same arguments as in Lemma 6.6 (see also [**31**]) combined with (6.10). The same conclusion can be drawn for the case $\xi_s \in \left\{\rho_q z^*_{r,q,\delta}\right\}$ since the two sets plays symmetric roles. The proof of the lemma is complete.

$\square$

According to Chui-Zhong's theorem [**31**] the family $\mathcal{X} = \{\{\xi_{n,j}\}_{j=0}^{n}\}_{n \geq 0}$ of the points on the unit circle is an L^α Marcinkiewicz-Zygmund family if and only if it is uniformly separated and there exist a constant K_α such that

$$(6.11) \quad \Big(\frac{1}{|I|}\int_I |F_n(e^{i\theta})|^\alpha d\theta\Big)^{\frac{1}{\alpha}} \Big(\frac{1}{|I|}\int_I |F_n(e^{i\theta})|^{-\frac{\alpha}{\alpha-1}} d\theta\Big)^{\frac{1}{\alpha}} \leq K_\alpha$$

For every subarc I of the unit circle and every $n \geq 0$.
We notice that the fact that the family is uniformly separated insure that this family generates a Carlson measure, and it is turn out that the second condition (6.11) is well-known as A_α condition in the setting of the BMO spaces (Bounded Mean Oscillation) [**49**, p.215]. We remind that locally integrable positive function w satisfy A_α condition if

$$(6.12) \quad \sup_I \Big(\frac{1}{|I|}\int_I w(x)dx\Big)\Big(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{\alpha-1}} dx\Big)^{\alpha-1} < \infty.$$

It turn out that in the case $p = 2$ the condition (6.12) is equivalent to the following Helson-Szegö condition:

Helson-Szegö condition. There are real-valued function $u, v \in L^\infty(\mathbb{T})$ such that

$$\|v\|_\infty < \frac{\pi}{2} \text{ and } w = e^{u+\widetilde{v}}, \text{ (HS)}$$

where $\widetilde{v}$ is the conjugate function of v.

For the proof of the equivalence of (6.12) when $p = 2$ and (HS), we refer to [**49**, p.246-259]. Therein, the reader can found also the proof of the prediction Helson-Szegö's theorem related to (HS) [**56**].

Now, according to the equivalence of (6.12) when $p = 2$ and (HS), Marzo and Seip [**76**] observe that in order to prove that the condition (6.11) holds it suffices to establish that the following uniform Helson-Szegö condition holds:

Uniform Helson-Szegö condition. There exist sequence u_n and v_n of real-valued function in $L^\infty(\mathbb{T})$ such that

$$\sup_n \|v_n\|_\infty < \frac{\pi}{2}, \ \sup_n \|u_n\|_\infty < +\infty \text{ and } |F_n|^2 = e^{u_n+\widetilde{v_n}},$$

where $\widetilde{v_n}$ is the conjugate function of v_n.

We are going to prove that the uniform Helson-Szegö condition holds. Let $\kappa > 0$ and $n = 2q - 1$. We claim first that we have

$$|F_n(e^{i\theta})|^2 = e^{u_{n,\kappa}(\theta)}|F_{n,\kappa}(e^{i\theta})|^2,$$

where

$$F_{n,\kappa}(z) = \prod_{r=0}^{n}\Big(1 - \rho_{n,\kappa}\overline{z_{r,n+1,\delta}}z\Big) \text{ and } \rho_{n,\kappa} = \max\Big\{\frac{1}{2}, 1 - \frac{\kappa}{n+1}\Big\}.$$

Indeed, the Mahler measure of the fonction $\phi_{n,\kappa}(\theta) \stackrel{\text{def}}{=} \dfrac{F_n(e^{i\theta})^2}{F_{n,\kappa}(e^{i\theta})^2}$ verify

$$M(|\phi_{n,\kappa}|) = \prod_{r=0}^{n}\Bigg(\frac{M\big(1 - \rho_n\overline{z_{r,n+1,\delta}}e^{i\theta}\big)}{M\big(1 - \rho_{n,\kappa}\overline{z_{r,n+1,\delta}}e^{i\theta}\big)}\Bigg) = 1,$$

by Proposition 2.5 combined with the well-know Jensen formula. We further have

$$\begin{aligned}\frac{1-\rho_n\overline{z_{r,n+1,\delta}}e^{i\theta}}{1-\rho_{n,\kappa}\overline{z_{r,n+1,\delta}}e^{i\theta}} &= \big(1-\rho_n\overline{z_{r,n+1,\delta}}e^{i\theta}\big)\Big(\sum_{l=0}^{+\infty}\rho_{n,\kappa}^l\overline{z_{r,n+1,\delta}}^l e^{il\theta}\Big)\\ &= 1+\sum_{l=0}^{+\infty}\rho_{n,\kappa}^{l-1}\overline{z_{r,n+1,\delta}}^l\big(\rho_{n,\kappa}-\rho_n\big)e^{il\theta}.\end{aligned}$$

Therefore $\phi_{n,\kappa}$ is in H^1 and $\log|\phi_{n,\kappa}|$ is integrable. Put

$$G(z)=\exp\Big(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|\phi_{n,\kappa}(\theta)|d\theta\Big).$$

Then G is an analytic function in the unit disc $\mathbb{D}$ and $\big|G\big|=e^{u_{n,\kappa}}$, where $u_{n,\kappa}$ is the Poisson integral of $\log(|\phi_{n,\kappa}|)$, that is, $u_{n,\kappa}(re^{i\theta})=P_r*\log(|\phi_{n,\kappa}|)$ where P_r is the Poisson kernel and $*$ is the convolution operator. By Fatou theorem [**57**, p.34], $|G|=e^{u_{n,\kappa}}=|\phi_{n,\kappa}|$ almost everywhere on the unit circle $\mathbb{T}$. We further have

$$\begin{aligned}u_{n,\kappa}(\theta) &= 2\mathrm{Re}\Big(\mathrm{Log}\big(F_n(\theta)\big)-\mathrm{Log}\big(F_{n,\kappa}(\theta)\big)\Big)\\ &= 2\mathrm{Re}\Big(\sum_{r=0}^{n}\Big(\mathrm{Log}\Big(1-\rho_n\overline{z_{r,n+1,\delta}}e^{i\theta}\Big)-\mathrm{Log}\Big(1-\rho_{n,\kappa}\overline{z_{r,n+1,\delta}}e^{i\theta}\Big)\Big)\Big)\\ &= 2\mathrm{Re}\Big(\sum_{r=0}^{n}\Big(\sum_{l=1}^{+\infty}\frac{\rho_{n,\kappa}^l-\rho_n^l}{l}\overline{z_{r,n+1,\delta}}^l e^{il\theta}\Big)\Big),\end{aligned}$$

where Log is *the principal value of the logarithm*. Writing

$$u_{n,\kappa}(\theta)=2\mathrm{Re}(I+II),$$

where

$$I=\sum_{r=0}^{q-1}\sum_{l=1}^{+\infty}\frac{\rho_{n,\kappa}^l-\rho_n^l}{l}\overline{z_{r,q}}^l e^{il\theta},\ \text{and}\ II=\sum_{r=0}^{q-1}\sum_{l=1}^{+\infty}\frac{\rho_{n,\kappa}^l-\rho_n^l}{l}\overline{z_{r,q}}^l e^{-2i\pi\frac{l\delta}{2q}}e^{il\theta}.$$

It follows that

$$I=\sum_{l=1}^{+\infty}\frac{\rho_{n,\kappa}^{lq}-\rho_n^{lq}}{l}e^{ilq\theta},\ \text{and}\ II=\sum_{l=1}^{+\infty}\frac{\rho_{n,\kappa}^{lq}-\rho_n^{lq}}{l}e^{-2il\pi\frac{\delta}{2}}e^{ilq\theta},$$

since

$$\sum_{r=0}^{q-1} \overline{z_{r,q}}^{l} = \begin{cases} q, & \text{if } l \in q\mathbb{Z}; \\ 0, & \text{if not.} \end{cases}$$

We can thus write

$$\begin{aligned} \left|u_{n,\kappa}\right| &\leq |I| + |II| \\ &\leq 2\Big(\sum_{l=1}^{+\infty} \frac{\rho_{n,\kappa}^{lq}}{l} + \sum_{l=1}^{+\infty} \frac{\rho_{n}^{lq}}{l}\Big) \\ &\leq 2\Big(\log\Big(\frac{1}{1-\rho_{n,\kappa}^{q}}\Big) + \log\Big(\frac{1}{1-\rho_{n}^{q}}\Big)\Big), \\ &\leq \frac{2}{1-e^{-\frac{\kappa}{2}}} + \frac{2}{1-e^{-\frac{1}{2}}} \end{aligned}$$

since $\log(x) \leq x$ for any $x > 0$, and $\log(1-x) \leq -x$ for $0 \leq x < 1$. We thus conclude that

$$\sup_n \left\|u_{n,\kappa}\right\|_{\infty} < +\infty,$$

and the proof of the claim is complete.

We move now to construct the functions v_n. For that, we start by proving the following lemma

Lemma 6.8. *Let $F(z) = (1 - rz_0 z)$, with $0 < r < 1$ and $z_0 = e^{i\theta_0}$. Then $|F|^2(e^{i\theta}) = e^{\widetilde{v}}$, where $\widetilde{v}$ is the conjugate function of the function v given by*

$$v(\theta) = P_r * \mathbb{1}_{[0,\theta]}(\theta_0) - \theta - c, \tag{6.13}$$

and c is any suitable constant.

Proof. Obviously F^2 is an outer function since the zeros of F^2 are out of the disc $\mathbb{D}$. We further have $F^2(0) = 1$. Whence $|F|^2 = e^{\widetilde{v}}$, where $\widetilde{v}$ is the conjugate function of the function v given by (6.13).

Indeed, for any θ, we have

$$\begin{aligned}
\mathrm{Log}\big(F(e^{i\theta})\big) &= -\sum_{n=1}^{+\infty}\frac{r^n}{n}e^{in(\theta-\theta_0)} \\
&= -\sum_{n=1}^{+\infty}\frac{r^n}{n}\cos\big(n(\theta-\theta_0)\big)+i\sum_{n=1}^{+\infty}\frac{r^n}{n}\sin\big(n(\theta-\theta_0)\big), \qquad (6.14)
\end{aligned}$$

where Log is *the principal value of the logarithm*. We further have

$$\begin{aligned}
v(\theta) &= P_r*\mathbb{1}_{[0,\theta]}(\theta_0)-\theta-c=\int_0^\theta P_r(\theta_0-t)dt-\theta-c \\
&= \int_0^\theta P_r(t-\theta_0)dt-\theta-c,
\end{aligned}$$

Since P_r is an even function. But

$$\begin{aligned}
\int_0^\theta P_r(t-\theta_0)dt &= \int_0^\theta \sum_{n=-\infty}^{+\infty} r^{|n|}e^{in(t-\theta_0)}dt \\
&= \sum_{n\neq 0}\frac{r^{|n|}}{in}e^{in(\theta-\theta_0)}-\sum_{n\neq 0}\frac{r^{|n|}}{in}e^{-in\theta_0}+\theta, \qquad (6.15)
\end{aligned}$$

by (6.2). Consequently

$$v(\theta)=\sum_{n\neq 0}\frac{r^{|n|}}{in}e^{in(\theta-\theta_0)}-\sum_{n\neq 0}\frac{r^{|n|}}{in}e^{-in\theta_0}-c,$$

and

$$\begin{aligned}
\widetilde{v}(\theta) &= -i\sum_{n\neq 0}\frac{n}{|n|}\frac{r^{|n|}}{in}e^{in(\theta-\theta_0)}, \\
&= -2\sum_{n=1}^{+\infty}\frac{r^n}{n}\cos\big(n(\theta-\theta_0)\big) \qquad (6.16)
\end{aligned}$$

Combining (6.14) with (6.16), we obtain

$$\mathrm{Re}\Big(\mathrm{Log}(F^2(e^{i\theta}))\Big)=\widetilde{v}(\theta),$$

which gives

$$|F^2|=e^{\widetilde{v}},$$

since for any analytic function g, we have

$$|e^g| = e^{\mathrm{Re}\big(g\big)},$$

and the proof of the lemma is complete. $\square$

We now apply Lemma 6.13 to write

$$|F_{n,\kappa}(e^{i\theta})|^2 = e^{\widetilde{v_{n,\kappa}}(\theta)},$$

where

$$v_{n,\kappa}(\theta) = \sum_{j=0}^{n} \int_0^{\theta} P_{\rho_{n,\kappa}}\big(\theta_{n,j} - t\big)dt - (n+1)\theta - c, \tag{6.17}$$

$$\theta_{n,j} = \begin{cases} 2\pi\frac{j}{2q}, & \text{if } j \text{ is even;} \\ 2\pi\big(\frac{j-1}{2q} + \frac{\delta}{q}\big), & \text{if } j \text{ is odd,} \end{cases}$$

and c is any suitable constant. Taking

$$c = \sum_{j=0}^{n} \int_{-2\pi\gamma_{n,j}}^{0} P_{\rho_{n,\kappa}}\Big(\Big(2\pi\frac{j-1}{2q}\Big) - t\Big)dt,$$

with

$$\gamma_{n,j} = \begin{cases} 0, & \text{if } j \text{ is even;} \\ 2\pi\frac{\delta}{q}, & \text{if } j \text{ is odd.} \end{cases}$$

We can rewrite (6.17) as

$$v_{n,\kappa}(\theta) = \sum_{j=0}^{n} \int_0^{\theta-2\pi\gamma_{n,j}} P_{\rho_{n,\kappa}}\big(\theta_{n,j} - t\big)dt - (n+1)\theta, \tag{6.18}$$

since, for any odd j, we have

$$\begin{aligned} \int_0^{\theta} P_{\rho_{n,\kappa}}(\theta_{n,j} - t)dt &= \int_0^{\theta} P_{\rho_{n,\kappa}}\Big(\Big(2\pi\Big(\frac{j-1}{2q}\Big) + \gamma_{n,j}\Big) - t\Big)dt, \\ &= \int_{-2\pi\gamma_{n,j}}^{\theta-2\pi\gamma_{n,j}} P_{\rho_{n,\kappa}}\Big(\Big(2\pi\Big(\frac{j-1}{2q}\Big)\Big) - t\Big)dt \end{aligned}$$

Again writing

$$v_{n,\kappa}(\theta) = I + II,$$

where

$$I=\sum_{r=0}^{q-1}\int_0^{\theta}P_{\rho_{n,\kappa}}\Big(\frac{2\pi j}{q}-t\Big)dt-q\theta \text{ and } II=\sum_{r=0}^{q-1}\int_0^{\theta-\frac{2\pi\delta}{q}}P_{\rho_{n,\kappa}}\Big(\frac{2\pi j}{q}-t\Big)dt-q\theta.$$

We thus need to estimate $|I|$ and $|II|$. But, by the same reasoning as above, it is easy to check that

$$I=\sum_{l\neq 0}\rho_{n,\kappa}^{lq}\frac{1-e^{-ilq\theta}}{il} \text{ and } II=\sum_{l\neq 0}\rho_{n,\kappa}^{lq}\frac{1-e^{-ilq(\theta-\frac{2\pi\delta}{q})}}{il}-2\pi\delta.$$

Consequently, we get

$$|I|\leq 4\sum_{l\geq 1}\frac{\rho_{n,\kappa}^{lq}}{l}=-4\log(1-\rho_{n,\kappa}^{q}).$$

Whence

$$|I|\lesssim -4\log\Big(1-e^{-\frac{\kappa}{2}}\Big).$$

It is still to estimate $|II|$. In the same manner it can be seen that

$$|II|\lesssim 2\pi\delta+4\sum_{l\geq 1}\frac{\rho_{n,\kappa}^{lq}}{l}\leq 2\pi\delta-4\log\Big(1-e^{-\frac{\kappa}{2}}\Big),$$

and by choosing κ sufficiently large and $\delta<\frac{1}{8}$, we obtain

$$\sup_n \|v_{n,\kappa}(\theta)\|_{\infty}<\frac{\pi}{2}.$$

From this we conclude that the uniform Helson-Szegö condition holds for $\alpha=2$.
For the case $1<\alpha\neq 2$. Assuming $\delta<\frac{1}{8\beta}$ where $\beta=\max\big\{\alpha,\frac{\alpha}{\alpha-1}\big\}$, one may apply the standard argument from the H^p theory combined with the Hölder inequality and Lemma 2 from [**76**] to conclude that the uniform Helson-Szegö condition holds.

Now, let $0<\alpha<2$ and $0<\delta<\frac{1}{4\beta}$, with $\beta=\frac{\alpha}{\alpha-1}$. By Lemma 6.7 the family $\mathcal{Z}=\Big\{\big\{\rho_q z_{r,q}\big\}\Big\}_{q\geq 0}\cup\Big\{\big\{\rho_q z^*_{r,q,\delta}\big\}\Big\}_{q\geq 0}$ is uniformly separated.

We can thus write

$$\int \Big||P_q|^2-1\Big|^{\alpha} d\theta$$
$$(6.19)\ \leq C_{\alpha,\delta}\Big(\frac{1}{2q}\sum_{r=0}^{q-1}\Big||P_q\big(z_{r,q}\big)|^2-1\Big|^{\alpha}+\frac{1}{2q}\sum_{r=0}^{q-1}\Big||P_q\big(z_{r,q,\delta}^*\big)|^2-1\Big|^{\alpha}\Big),$$

where

$$C_{\alpha,\delta}=\frac{2C_\gamma}{\gamma^2.\frac{\delta}{\sqrt{1+\delta^2}}}=\frac{2C_\gamma\sqrt{1+\delta^2}}{\gamma^2.\delta}.$$

The computation of constant $C_{\alpha,\delta}$ can be found in [**40**, p.153]. Therein, by appealing to the duality argument, it is shown that for any $w=(w_j)\in \ell^\alpha$, there exist $g\in H^\alpha$ such that for some $f\in H^\beta$, with $\big\|f\big\|_\beta=1$, and β is the conjugate of α, one can assert

$$\big\|g\big\|_\alpha \leq \frac{\sqrt{1+\delta^2}}{\gamma^2.\delta}\|w\|_\alpha\Big(\int_{\mathbb{D}}|f(z)|^\beta d\mu_{\mathcal{Z}}+\int_{\mathbb{D}}|f(z)|^\beta d\mu_{\mathcal{Z}^*}\Big),$$

where

$$\mu_{\mathcal{Z}}=\sum_{q=3}^{+\infty}\Big(\sum_{r=0}^{q-1}\big(1-|\rho_q z_{r,q}|\big)\delta_{z_{r,q}},\ \text{and}\ \mu_{\mathcal{Z}^*}=\sum_{q=3}^{+\infty}\Big(\sum_{r=0}^{q-1}\big(1-|\rho_q z_{r,q,\delta}^*|\big)\delta_{z_{r,q}}.$$

But the measures $\mu_{\mathcal{Z}}$ and $\mu_{\mathcal{Z}^*}$ are a Carleson measures. Therefore,

$$\int_{\mathbb{D}}|f(z)|^\beta d\mu_{\mathcal{Z}}+\int_{\mathbb{D}}|f(z)|^\beta d\mu_{\mathcal{Z}^*}\leq 2C_\gamma\|f\|_\beta=2C_\gamma,$$

where

$$C_\gamma=\frac{2}{\gamma^4}\big(1-2\log(\gamma)\big).$$

An alternative proof can be found in [**57**, p.195-202]. The reader may notice that the proof of Theorem F in [**33**] can be drawn from the above

proof. We further have

$$\begin{aligned}
&\Big|\Big(\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z_{r,q}\big)|^2-1\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}-\Big(\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z^*_{r,q,\delta}\big)|^2-1\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}\Big|\\
&\leq \Big(\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z_{r,q}\big)|^2-\big|P_q\big(z^*_{r,q,\delta}\big)|^2\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}\\
&\leq 2^{\frac{1}{\alpha}}\Big(\frac{1}{2q}\sum_{r=0}^{2q-1}\Big||P_q\big(z_{r,2q}\big)|^2-\big|P_q\big(z^*_{r,2q,2\delta}\big)|^2\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}\\
&\leq c_\alpha\frac{\delta}{2q}\Big(\int\Big|\Big(|P(\theta)|^2\Big)'\Big|^{\alpha}d\theta\Big)^{\frac{1}{\alpha}}.
\end{aligned}$$

The second inequality is due to the simple fact that $z_{2r,2q}=z_{r,q}$ and $z^*_{2r,2q,2\delta}=z^*_{r,2q,\delta}$, and the third inequality follows by the same arguments as in the proof of Lemma 12 and Theorem 9 from [**83**]. Applying Bernstein-Zygmund inequality (Theorem 6.2) combined with the classical Marcinkiewicz-Zygmund inequalities, we obtain

$$\begin{aligned}
&\Big|\Big(\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z_{r,q}\big)|^2-1\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}-\Big(\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z^*_{r,q,\delta}\big)|^2-1\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}\Big|\\
&\leq c_\alpha\delta\Big(\int|P(\theta)|^{2\alpha}d\theta\Big)^{\frac{1}{\alpha}}\\
&\leq c'_\alpha\delta\Big(\frac{1}{q}\sum_{r=0}^{q-1}|P(z_{r,q})|^{2\alpha}\Big)^{\frac{1}{\alpha}}.
\end{aligned}$$

The reader should notice that the constant c'_α depend only on α. If follows that

$$\begin{aligned}
&\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z^*_{r,q}\big)|^2-1\Big|^{\alpha}\\
&\leq\Big(\Big(\frac{1}{q}\sum_{r=0}^{q-1}\Big||P_q\big(z_{r,q}\big)|^2-1\Big|^{\alpha}\Big)^{\frac{1}{\alpha}}+c'_\alpha\delta\Big(\frac{1}{q}\sum_{r=0}^{q-1}|P(z_{r,q})|^{2\alpha}\Big)^{\frac{1}{\alpha}}\Big)^{\alpha}.
\end{aligned}$$

Combined these inequalities with Lemma 6.4, we can rewrite (6.19) as

$$\int\Big|\big|P_q\big|^2-1\Big|^{\alpha}d\theta\leq C_{\alpha,\delta}\Big(\frac{1}{2}I_{\alpha,p}+\frac{1}{2}\Big(I^{\frac{1}{\alpha}}_{\alpha,p}+c'_\alpha\delta J^{\frac{1}{\alpha}}_{\alpha,p}\Big)^{\alpha}\Big),$$

where

$$\begin{aligned} I_{\alpha,p} &= \frac{p^\alpha}{q}+\Big(\frac{q-1}{q}\Big)\Big(\frac{p}{p+1}-1\Big), \text{ and} \\ J_{\alpha,p} &= \frac{(p+1)^\alpha}{q}+\Big(\frac{q-1}{q}\Big)\Big(\frac{p}{p+1}\Big) \end{aligned}$$

Letting $q \longrightarrow +\infty$, we obtain

$$\lim_{q\longrightarrow+\infty}\int\Big|\big|P_q\big|^2-1\Big|^\alpha d\theta \leq C_{\alpha,\delta}\frac{{c'}_\alpha^\alpha}{2}\delta^\alpha = \frac{C_\gamma}{\gamma^2}\sqrt{1+\delta^2}{c'}_\alpha^\alpha\delta^{\alpha-1},$$

and by letting $\delta \longrightarrow 0$, we conclude that

$$\lim_{q\longrightarrow+\infty}\int\Big|\big|P_q\big|^2-1\Big|d\theta = 0,$$

Since

$$\int\Big|\big|P_q\big|^2-1\Big|d\theta \leq 2, \text{ and } \ \alpha > 1.$$

Hence the sequence of polynomials $(P_q)_{q\in\mathbb{N}}$ is L^1-flat.

Therefore, by appealing to Proposition 4.2, we deduce that the sequence of polynomials $(P_q(z))$ is almost everywhere flat over some subsequence. Thus the proof of Theorem 2.1 is complete.

Theorem 2.2 follows from Theorem 2.1 combined with Proposition 4.2 and Lemma 6.6. Finally, by (3.2), we deduce that the spectral type σ of the rank one map constructed in Theorem 2.2 verify

$$M\Big(\frac{d\sigma}{dz}\Big) = \prod_{j=0}^{+\infty} M(P_j^2) > 0.$$

Whence,

$$M(P_j) \xrightarrow[j\to+\infty]{} 1.$$

This finishes the proof. For more details on the construction of rank one map in Theorem 2.2, we refer the reader to [**3**] and [**4**].

Remarks. Obviously, as in the proof given by Zygmund in [**102**, p.29, Chap X], in our proof of Theorem 6.3, we take advantage of the

following classical identity [**102**, p.35, Chap II]

$$\frac{1}{d}\sum_{j=0}^{d-1} e^{\frac{2\pi ijk}{d}} = \begin{cases} 0 & \text{if } d \nmid k \\ 1 & \text{if } d \mid k, \end{cases}$$

for any $d, k \geq 1$.

The reader may notice that there is some analogies between our proof and the Fast Fourier Transform algorithm (FFT). We refer the reader to [**98**] for more details on the FFT.

Applying Carleson interpolation theory, one can prove that for any $p > 0$, there is a constant $C_p > 0$ such that, for any polynomial P of degree less than n,

$$\frac{C_p^{-1}}{4n}\sum_{j=0}^{4n-1} \left|P(e^{2\pi i\frac{j}{2n}})\right|^p \leq \left\|P\right\|_p^p.$$

An alternative proof can be found in [**85**]. Besides this, Marcinkiewicz and Zygmund proved [**75**] that for any $p \geq 1$ and for any polynomial P of degree less or equal than n, we have

$$\left(\frac{1}{2n}\sum_{j=0}^{2n-1} \left|P(e^{2\pi i\frac{j}{2n}})\right|^p\right)^{\frac{1}{p}} \leq \big(p\pi + 1\big)^{\frac{1}{p}}\left\|P\right\|_p.$$

To the best of this author's knowledge, the explicit constant for the case $p = 0$ seems not to be known. Nevertheless, in the case of the classical Riesz product, if we consider the polynomial

$$P(\theta) = 1 + \alpha\cos(n\theta),$$

where α is non-negative number less than 1. Then

$$\begin{aligned} M_{d\omega_{4n+1}}(P) &= \exp\left(\frac{1}{4n}\sum_{j=0}^{4n-1}\log\left(\left|P(e^{2\pi i\frac{j}{2n}})\right|\right)\right) \\ &\leq M_{dz}(P). \end{aligned} \tag{6.20}$$

This can be proved as follows.

Following [**60**], we put

$$P(\theta) = |Q(e^{i\theta})|^2,$$

where

$$Q(z) = \frac{1+az^n}{1+a^2}, \text{ with } a = \frac{\alpha}{1+\sqrt{1-\alpha^2}}.$$

It is easily seen that $Q(z)$ does not vanish on the disc $D \stackrel{\text{def}}{=} \{|z| \leq 1\}$. We thus get that the function $\log\left(\left|Q(e^{i\theta})\right|^2\right)$ is harmonic. Applying the mean property, we obtain

$$\log\left(\left|Q(0)\right|^2\right) = \frac{1}{2\pi}\int_0^{2\pi} \log(\left|Q(e^{i\theta})\right|^2) d\theta. \tag{6.21}$$

Rewriting (6.21), we see that

$$M(P) = \frac{1}{1+a^2}.$$

Now, any easy computation shows that

$$P\left(e^{2\pi i \frac{j}{2n}}\right) = 1 + \alpha(-1)^j,$$

for $j = 0, \cdots, 2n-1$.
Whence

$$M_{d\omega_{4n+1}}(P) = \sqrt{1-\alpha^2}.$$

Obviously

$$\sqrt{1-\alpha^2} \leq \frac{1}{1+a^2}.$$

We conclude that (6.20) holds.

This leads us to ask.

Questions.

(1) Can one prove or disapprove that $C_p^{\frac{1}{p}}$ converge to 1 as $p \longrightarrow 0$.
(2) Let S_p be a family of Singer sets, p is a prime number and consider the sequence of polynomials

$$P_q(z) = \frac{1}{\sqrt{|S_p|}} \sum_{s \in S_q} z^s, \ |z| = 1.$$

Can one prove or disapprove that the sequence of the Mahler measure of P_q converge to 1.

(3) Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system where μ is a finite measure. Can one prove or disapprove that there exist a Borel set A with $\mu(A) > 0$ such that for μ-almost all $x \in X$,

$$\int \Big|\frac{1}{\sqrt{N\mu(A)}}\sum_{j=0}^{N-1} \mathbb{1}_A(T^j x) z^j\Big| dz \xrightarrow[N\to+\infty]{} 1.$$

(4) In the same setting as in the previous question, can one prove or disapprove that for any measurable f with values ± 1, for μ-almost all $x \in X$, we have

$$\limsup_{N \longrightarrow +\infty} \int \Big|\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} f(T^j x) z^j\Big| dz < 1.$$

As mentioned in introduction, this problem can be linked to the annealed and quenched business.

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